

On quantum percolation in finite regular graphs

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Abstract

The aim of this paper is twofold. First, we study eigenvalues and eigenvectors of the adjacency matrix of a bond percolation graph when the base graph is finite and well approximated locally by an infinite regular graph. We relate quantitatively the empirical measure of the eigenvalues and the delocalization of the eigenvectors to the spectrum of the adjacency operator of the percolation on the infinite graph. Secondly, we prove that percolation on an infinite regular tree with degree at least 3 preserves the existence of an absolutely continuous spectrum if the removal probability is small enough. These two results are notably relevant for bond percolation on a uniformly sampled regular graph or a Cayley graph with large girth.

1 Introduction

In the seminal work [5], Anderson has studied the transport properties of a quantum particle on a regular lattice in the presence of random impurities. Shortly after, De Gennes, Lafore and Millot [16, 17] have proposed to study the transport properties on a randomly disordered lattice. The latter is now usually referred as quantum percolation and only results on the density of states are currently available [32, 15, 45, 13], see notably [6, 39] for survey and references. Mathematically, it amounts to study the regularity of the spectral measures of Laplacian-type operators of the disordered lattice. Since the landmark result of Klein [33], perturbation methods have been used to study random operators on infinite regular trees (Bethe lattice). The spectrum of the Laplacian operator of a nearly-regular Galton-Watson tree without leaves has notably been studied recently by Keller [30]. In parallel, Kottos and Smilansky [36] have suggested that spectral statistics of some finite graphs are in good agreement with random matrix theory and the predictions of quantum chaos. In [42, 43], Terras has also discussed the connections between finite quantum chaos, spectrum of graphs and random matrix theory. On large finite regular graphs, local spectral distribution and delocalization of eigenvectors have notably been studied in [37, 19, 14, 44, 4, 27].

In this paper, in the spirit of De Gennes, Lafore and Millot, we study eigenvalues and eigenvectors of the adjacency matrix of finite percolation graphs. More precisely, we consider a large

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graph G which is well approximated locally by an infinite regular graph, say Γ . We keep each edge of G independently with probability p , remove it otherwise, and consider the adjacency matrix of the corresponding randomly diluted graph, denoted by $\text{perc}(G, p)$. We relate quantitatively some notion of regularity of the spectral measure and the delocalization of the eigenvectors of $\text{perc}(G, p)$ to the spectrum of the adjacency operator of the percolation on Γ , $\text{perc}(\Gamma, p)$. These finite volumes corrections are stated in a general framework and have nearly the same order than the recent results on d -regular graphs and $p = 1$, [14, 44, 4, 27]. Using a perturbation technique, we also complement the result of Keller [30] on supercritical Galton-Watson trees to the situation where the tree may have leaves. Compared to the above mentioned results, an important new difficulty in quantum percolation is the concomitant presence of point and continuous spectrum, see [32, 15] or [8, §3.2].

This paper is organized as follows. The remainder of the introduction presents the main definitions and the main results. Section 2 analyses the spectrum of Galton-Watson trees whose offspring distribution is close to deterministic. Section 3 presents basic resolvent bounds. Finally, in Sections 4-5 we apply these bounds to finite percolation graphs.

1.1 Adjacency operator and spectral measures

Let $G = (V, E)$ be a locally finite (undirected) graph, that is a simple graph such that all vertices have a finite degree. We may consider the Hilbert space

$$\ell^2(V) = \left\{ \psi: V \rightarrow \mathbb{C}, \sum_{x \in V} |\psi(x)|^2 < \infty \right\},$$

with inner product $\langle \phi, \psi \rangle = \sum_{x \in V} \overline{\phi(x)} \psi(x)$. Denote by $\ell_0^2(V) \subseteq \ell^2(V)$ the dense subspace of finitely supported functions, and by $(e_x, x \in V)$ the canonical orthonormal basis of $\ell^2(V)$, i.e. e_x is the coordinate function $y \in V \mapsto \mathbf{1}(x = y)$. The adjacency operator A of G is the linear operator over $\ell^2(V)$ whose domain is $\ell_0^2(V)$ and whose action on the basis vector $e_x, x \in V$ is:

$$Ae_x = \sum_{y: xy \in E} e_y.$$

Note that $Ae_x \in \ell^2(V)$ since G is locally finite. Moreover, for all $x, y \in V$,

$$\langle Ae_x, e_y \rangle = \mathbf{1}\{xy \in E\} = \langle Ae_y, e_x \rangle.$$

Hence, the operator A is symmetric, and we may ask about its (essential) self-adjointness, that is the self-adjointness of its closure. If the degrees of vertices of G are uniformly bounded then A is a bounded operator and hence self-adjoint on $\ell^2(V)$. We will pay a special attention to adjacency operators of tree. A sufficient condition for self-adjointness was given in [12, Proposition 3].

Recall that if A is essentially self-adjoint then the spectral measure at vector e_x (or at vertex $x \in V$) is well-defined. It is the unique probability measure on \mathbb{R} , denoted by $\mu_G^{e_x}$, such that for all

integer $k \geq 0$,

$$\int x^k d\mu_G^{e_x} = \langle e_x, A^k e_x \rangle. \quad (1)$$

Note that the right hand side is the number of closed paths in G of length k starting at x . We will say that A has non-trivial absolutely continuous spectrum if there exists $x \in V$ such that $\mu_G^{e_x}$ has an absolutely continuous part of positive mass. Also, A has a purely absolutely continuous spectrum on an interval (a, b) if for all $x \in V$, $\mu_G^{e_x}$ is absolutely continuous on (a, b) .

If G is a finite with $|V| = n$ vertices, we define classically the spectral measure as

$$\mu_G = \frac{1}{n} \sum_{k=1}^n \delta_{\lambda_k},$$

where $\lambda_1, \dots, \lambda_n$ are the real eigenvalues of A . It is straightforward to check that μ_G can also be written as the spatial average of the spectral measures at the vertices :

$$\mu_G = \frac{1}{|V|} \sum_{v \in V} \mu_G^{e_v}.$$

Motivated by the Benjamini-Schramm local graph topology, see [3], we will be interested by random rooted graphs (G, o) , random graphs with a distinguished vertex $o \in V$, the root. In which case, the expected spectral measure $\mathbb{E}\mu_G^{e_o}$ will play an important role. When the law of (G, o) is unimodular (that is satisfies a specific mass transport principle), $\mathbb{E}\mu_G^{e_o}$ can be interpreted as a density of states. We refer to the introduction of [13] and [8] for more details.

If $G = (V, E)$ is a graph, we will denote by $\text{perc}(G, p)$ the random graph with vertex set V and edge set $E' \subset E$ obtained by keeping each edge of E independently with probability p and removing it otherwise.

To motivate the sequel, we now briefly argue that expected spectral measures of percolation graphs has typically a dense set of atoms on its support. Take $0 < p < 1$ and let $G = \text{perc}(\Gamma, p)$ be the percolation graph of some infinite graph Γ with uniformly bounded degrees (for example Γ is the lattice \mathbb{Z}^d or the infinite d -regular tree), then G will have finite connected components with probability one. The spectral measures at vertices belonging to these finite connected components will be pure point. More importantly, the spectral measure of some vertices on infinite connected components will also have non-trivial atomic parts. This is notably due to the presence of finite pending subgraphs, indeed, it is not hard to build localized eigenvectors associated to eigenvalues of the adjacency matrix of these finite pending subgraphs, for a detailed argument see [32, 15] or [8, §3.2].

1.2 Extended states in Galton-Watson trees

Let $P = (P_k)_{k \geq 0} \in \mathcal{P}(\mathbb{Z}_+)$ be a probability distribution on non-negative integers. A Galton-Watson tree with offspring distribution P (GW(P) tree for short) is the random rooted (T, o)

defined as follows. Let $\mathbb{N}_f = \cup_{k \geq 0} \mathbb{N}^k$ with $\mathbb{N}^0 = \{o\}$ be the set of finite sequences of integers and let $(N_x)_{x \in \mathbb{N}_f}$ be independent variables with common distribution P . The vertex set V of T is the subset of \mathbb{N}^f obtained iteratively as follows: the offspring of $x = (i_1, \dots, i_k) \in \mathbb{N}^k \cap V$ are $V_x = \{(i_1, \dots, i_k, \ell), 1 \leq \ell \leq N_x\}$. Proposition 7 in [12] asserts that if $\mathbb{E}N_o < \infty$ and A is the adjacency operator of T then with probability one, A is essentially self-adjoint.

In the specific case where $P = \delta_q$ is a Dirac mass at q , then T is the infinite q -ary tree. It is not hard to check (see forthcoming Section 2) that, in this case, $\mu_T^{e_o}$ is the Wigner semicircle distribution with radius $2\sqrt{q}$. More precisely, $\mu_T^{e_o}$ has density on $[-2\sqrt{q}, 2\sqrt{q}]$ given by

$$f_q(\lambda) = \frac{1}{2\pi q} \sqrt{4q - \lambda^2}.$$

Our first result asserts that if P is close enough to δ_q then the adjacency operator of T has an absolutely continuous part. For $p \geq 1$, if N has distribution P , the Wasserstein L^p -distance to the Dirac mass δ_q is given by

$$W_p(P, \delta_q) = \mathbb{E}|N - q|^p = \sum_{k=0}^{\infty} |k - q|^p P(k).$$

Theorem 1. *Let A be the adjacency operator of T , a $\text{GW}(P)$ tree. Let $q \geq 2$ be an integer. There exists $\varepsilon = \varepsilon(q) > 0$ such that if $W_1(P, \delta_q) < \varepsilon$, then A has a non-trivial absolutely continuous spectrum with positive probability. Moreover, if f denotes the density of the absolutely continuous part of the spectral measure at the root $\mu_T^{e_o}$ of A ,*

$$\lim_{P \xrightarrow{L_1} \delta_q} \int \mathbb{E}|f(\lambda) - f_q(\lambda)| d\lambda = 0. \quad (2)$$

Theorem 1 will be proved by using a technique first developed in Aizenman, Sims and Warzel [2]. For any $p > 1$ and $q \geq 2$, Keller [30] has proved that there exists some $\varepsilon' = \varepsilon'(p, q) > 0$ such that if $W_p(P, \delta_q) < \varepsilon'$ and $P(0) = 0$ then A has absolutely continuous spectrum with probability one. In particular, Theorem 1 complements, Keller's result when $P(0) \neq 0$. Theorem 1 has the following corollary on the density of states.

Corollary 2. *With the notation of Theorem 1, if $W_1(P, \delta_q) < \varepsilon$, the expected spectral measure $\mathbb{E}\mu_T^{e_o}$ has a non-trivial absolutely continuous part $\bar{f}(\lambda)d\lambda$ and*

$$\lim_{P \xrightarrow{L_1} \delta_q} \int |\bar{f}(\lambda) - f_q(\lambda)| d\lambda = 0. \quad (3)$$

As already mentioned, unimodular random rooted graphs plays a central role in Benjamini-Schramm local graph topology. Assume that P has a positive and finite first moment. The unimodular Galton-Watson tree with degree distribution P (UGW(P) tree for short) is the random

rooted (T, o) defined as above, where N_o has distribution P and for all other $x \in \mathbb{N}^f \setminus \{o\}$, N_x are independent with common distribution \hat{P} defined by

$$\hat{P}(k) = \frac{(k+1)P(k+1)}{\sum_{\ell} \ell P(\ell)}.$$

As its name suggests, the distribution $\text{UGW}(P)$ is unimodular, see [3]. Also, this distribution is the Benjamini-Schramm limit of numerous graph sequences. For example, if $q \geq 1$ and $P = \delta_{q+1}$, then $\hat{P} = \delta_q$ and $\text{UGW}(\delta_{q+1})$ is a Dirac mass at the infinite $(q+1)$ -regular tree. In this case, $\mu_T^{e_o}$ is the Kesten-McKay distribution, it has density on $[-2\sqrt{q}, 2\sqrt{q}]$ given by

$$\check{f}_q(\lambda) = \frac{(q+1)}{2\pi} \frac{\sqrt{4q - \lambda^2}}{(q+1)^2 - \lambda^2}.$$

Our next result adapts the above statements to unimodular Galton-Watson trees.

Theorem 3. *Let A be the adjacency operator of T , a $\text{UGW}(P)$ tree. Let $q \geq 2$ be an integer. There exists $\varepsilon = \varepsilon(q) > 0$ such that if $W_2(P, \delta_{q+1}) < \varepsilon$, then A has a non-trivial absolutely continuous spectrum with positive probability. Moreover, the conclusions (2) of Theorem 1 and (3) of Corollary 2 hold with f_q replaced by \check{f}_q and $P \xrightarrow{L_1} \delta_q$ by $P \xrightarrow{L_2} \delta_{q+1}$.*

We may apply the above theorem to bond percolation on the infinite $(q+1)$ -regular tree, say T_q . Then, the connected component of the root in $\text{perc}(T_q, p)$ has distribution $\text{UGW}(\text{Bin}(q+1, p))$. Recall that this connected component is infinite with positive probability if and only if $pq > 1$. If A is the adjacency operator of a $\text{UGW}(\text{Bin}(q+1, p))$ tree, in [13], it is proved that the density of states $\mathbb{E}\mu_{\text{perc}(T_q, p)}^{e_o}$ has a non-trivial continuous part if and only if $pq > 1$. We may define the quantum percolation threshold,

$$p_q = \sup\{p \geq 0 : A \text{ has no absolutely continuous spectrum with probability one}\}.$$

By unimodularity [3, Lemma 2.3], p_q is also equal to

$$p_q = \sup\{p \geq 0 : \mu_{\text{perc}(T_q, p)}^{e_o} \text{ has a trivial absolutely continuous part with probability one}\}.$$

We may also define a mean quantum percolation threshold,

$$\bar{p}_q = \sup\{p \geq 0 : \mathbb{E}\mu_{\text{perc}(T_q, p)}^{e_o} \text{ has a trivial absolutely continuous part}\}.$$

From what precedes, $1/q \leq \bar{p}_q \leq p_q \leq 1$. As a corollary of Theorem 3, we find

Corollary 4. *For any integer $q \geq 2$, we have $p_q < 1$.*

Note that due to the lack of monotonicity, it is not clear whether $p_q = p_q^*$ where $p_q^* = \inf\{p \leq 1 : \mu_{\text{perc}(T_q, p)}^{e_o} \text{ has a non-trivial absolutely continuous part with positive probability}\}$. It is also unknown if the strict inequality $\bar{p}_q < p_q$ holds or not.

1.3 Rates of convergence in percolation graphs

We now give quantitative finite size corrections on linear functions of the eigenvalues of percolation graphs. Let $\Gamma = (V(\Gamma), E(\Gamma))$ be a vertex transitive graph and G a finite graph on n vertices. For integer $h \geq 1$ and $v \in V(G)$, we denote by $(G, v)_h$ the subgraph of G spanned by the vertices which are at distance at most h from v . Also, $B_\Gamma(h)$ is the number of vertices of G such that $(G, v)_h$ is not isomorphic to $(\Gamma, o)_h$ where $o \in V(\Gamma)$.

For example, let $q \geq 2$ be an integer and assume further that G is a $(q+1)$ -regular graph. Then $B_{T_q}(h)$ is the number of vertices v in $V(G)$ such that $(G, v)_h$ is not a tree. Observe that if G has girth g (length of the shortest cycle), then $B_{T_q}(h) = 0$ for $h < g/2$. Also, if G is a uniformly sampled $(q+1)$ -regular graph on n vertices, then, for any $0 < \alpha < 1$, with probability tending to 1 as $n \rightarrow \infty$, $B_{T_q}(h) = n^{\alpha+o(1)}$ where $h = \lfloor (\alpha \log n)/(2 \log q) \rfloor$. This follows from known asymptotic on the number of cycles in random regular graphs, see [19, 38].

If $\varphi : \mathbb{R} \rightarrow \mathbb{R}$ has its derivative $\partial\varphi$ in $L^1(\mathbb{R})$, we set $\|\varphi\|_{\text{TV}} = \int |\partial\varphi(x)|dx$. The next statement will be a consequence of Jackson's approximation theorem.

Theorem 5. *Let G be a graph with n vertices and maximal degree d . For $p \in [0, 1]$, let $\mu = \mathbb{E}\mu_{\text{perc}(\Gamma, p)}^{e_o}$ and let φ be a C^k -function with $1 \leq k \leq 2h$. We have*

$$\left| \int \varphi d\mu_{\text{perc}(G, p)} - \int \varphi d\mu \right| \leq t + 2 \frac{B_\Gamma(h)}{n} \|\varphi\|_\infty + 2 \left(\frac{\pi d}{2} \right)^k \frac{(2h - k + 1)!}{(2h + 1)!} \|\partial^{(k)}\varphi\|_\infty,$$

with probability at least $1 - 2 \exp(-nt^2/(8\|\varphi\|_{\text{TV}}^2))$.

To understand better the above statement, we can apply it to the Cauchy-Stieltjes transform $\varphi_z(x) = 1/(x - z)$ which we will denote by

$$g_\mu(z) = \int \frac{d\mu(\lambda)}{\lambda - z}.$$

Roughly speaking, if $hB_\Gamma(h) = o(n)$, the Cauchy-Stieltjes transform converges as soon as $\Im(z) \geq C(\log h)/h$. More precisely, we will obtain for example the following corollary.

Corollary 6. *Let G be a graph with n vertices and maximal degree $d \geq 2$. For $p \in [0, 1]$, let $\mu = \mathbb{E}\mu_{\text{perc}(\Gamma, p)}^{e_o}$ and*

$$\delta \geq \frac{hB_\Gamma(h)}{n} \vee \frac{1}{h}. \tag{4}$$

Then, for any $z \in \mathbb{C}$ with $\Im(z) \geq 20d \log(2h)/h$,

$$\left| g_{\mu_{\text{perc}(G, p)}}(z) - g_\mu(z) \right| \leq \delta,$$

with probability at least $1 - 2 \exp(-n\delta^2/h^2)$.

If $p = 1$ and G is a d -regular graph, the above corollary recovers, up to the $O(\log h)$ factor for the lower bound on $\Im(z)$, statements in [19, 27], see also [14, 4]. It would be very interesting to extend Corollary 6 to some $z \in \mathbb{C}$ with $\Re(z)$ in the support of μ and $\Im(z) = o(1/h)$. Also in the bound (4), the term $1/h$ could be replaced by $1/h^k$ for any $k \geq 1$ by increasing suitably the constant 20. This would however not change much for the applications that will follow.

1.4 Regularity of the spectral measure in percolation graphs

As above, Γ is a vertex transitive graph. The next statement asserts that if $\mu_{\text{perc}(\Gamma, p)}$ has an absolutely continuous part and $B_\Gamma(h) = o(n)$ for some $h \gg 1$, then $\mu_{\text{perc}(G, p)}$ will also have some regularity property on intervals of scale $1/h$. The Lebesgue measure on \mathbb{R} is denoted by ℓ .

Theorem 7. *Let G be a graph with n vertices, maximal degree $d \geq 2$ and δ as in (4). For some $p \in (0, 1]$, assume that $\mu = \mathbb{E}\mu_{\text{perc}(\Gamma, p)}^{\epsilon_0}$ has an absolutely continuous part. Then, for any $\varepsilon > 0$, there exist positive constants c_0, c_1 and a deterministic closed set with $\ell(K) > 0$ such that with probability at least $1 - \delta^{-1}h^2 \exp(-n\delta^2/h^2)$, the following holds:*

(i) *If $\delta \leq c_0$, then for any $\lambda \in K$ and interval $I = (\lambda - \eta, \lambda + \eta)$ with $\eta \geq \eta_{\min} = c_1(\log h)/h$,*

$$\frac{\mu_{\text{perc}(G, p)}(I)}{\ell(I)} \leq c_1 \quad \text{and} \quad \frac{\mu_{\text{perc}(G, p)}(\bar{I})}{\ell(I)} \geq c_0 \frac{\eta_{\min}}{\eta}.$$

(ii) *If $\mu_s(\mathbb{R})$ is the total mass of the singular part of μ , we have $\mu_{\text{perc}(G, p)}(K^c) \leq 2\pi(\mu_s(\mathbb{R}) + \varepsilon + \delta)$.*

In the proof, the constants c_0, c_1 and the set K will depend on the absolutely continuous part of μ and ε in a rather straightforward manner. Note also that due to the 2π factor, statement (ii) is only useful when $\mu_s(\mathbb{R})$ is small enough. From Corollary 4, it is the case for example in $\text{perc}(T_q, p)$ when p is close to 1.

1.5 Weak delocalization in percolation graphs

We now turn to statements on delocalization of eigenvectors of $\text{perc}(G, p)$ when the adjacency operator of $\text{perc}(\Gamma, p)$ has a non-trivial continuous spectrum with positive probability. We will use a rather weak notion of delocalization in the underlying canonical basis.

Definition 8. *Let $(\rho, \varepsilon) \in [0, 1]^2$. A unit vector $\psi \in \mathbb{C}^n$ is (ρ, ε) -delocalized if there exists $S \subset [n]$ such that $\sum_{i \in S} |\psi(i)|^2 \geq \rho$ and for all $i \in S$, $|\psi(i)| \leq \varepsilon$.*

We also introduce some volumetric parameters of G . For $v \in V$, we set

$$N_h(G, v) = |V((G, v)_h)| \quad \text{and} \quad M_h(G) = \left(\frac{1}{|V|} \sum_{v \in V} N_h^2(G, v) \right)^{1/2}. \quad (5)$$

In words, $N_h(G, v)$ is the number of vertices at graph distance at most h from v and $M_h(G)$ is its quadratic average. Observe that if G has maximal degree d , then $N_h(G, v) \leq d(d-1)^{h-1}$.

Theorem 9. *Let G be a graph with n vertices, maximal degree $d \geq 2$ and δ as in (4). For some $p \in (0, 1]$, assume that $\mu_{\text{perc}(\Gamma, p)}^{e_o}$ has an absolutely continuous part with positive probability. Consider an orthogonal basis of eigenvectors of the adjacency matrix of $\text{perc}(G, p)$. Then, for any $\varepsilon > 0$, the following holds for some positive constants c_0, c_1, α, ρ (depending on $\varepsilon, d, \text{perc}(\Gamma, p)$). With probability at least $1 - \delta^{-1} h^2 \exp(-n\delta^2/(2h^2 M_h(G)^2))$, we have*

- (i) *If $\delta \leq c_0$, at least αn eigenvectors of $\text{perc}(G, p)$ are $\left(\rho, c_1 \sqrt{\frac{\log h}{h}}\right)$ -delocalized.*
- (ii) *If $\bar{\mu}_s(\mathbb{R})$ is the expected mass of the singular part of $\mu_{\text{perc}(\Gamma, p)}^{e_o}$, then we can take $\alpha = \rho = 1 - \sqrt{4\pi(\bar{\mu}_s(\mathbb{R}) + \varepsilon + \delta)}$.*

Again, the dependency of the constants in terms of the distribution of $\mu_{\text{perc}(\Gamma, p)}^{e_o}$ will be explicit. Our delocalization statement is weaker than other delocalization results obtained in [14, 4] on tree-like d -regular graphs and $p = 1$. Even, for this simpler class of graphs, it is an open problem to prove (ρ, ε) -delocalization with $\varepsilon = o(1/\sqrt{h})$. From Theorem 3, statement (ii) could be applied to $\text{perc}(T_d, p)$ and p close to 1. The proof of Theorem 9 will also rely on resolvent methods inspired by [22].

The proof of Theorems 7 and Theorem 9 will rely on resolvent methods introduced notably in the context of random matrices by Erdős, Schlein and Yau in [22].

2 Spectrum of Galton-Watson trees

2.1 Resolvent operator

Let $P = (P_k)_{k \geq 0} \in \mathcal{P}(\mathbb{Z}_+)$ with finite first moment and let T be a $\text{GW}(P)$ tree. As already pointed, with probability one, the adjacency operator A is essentially self-adjoint. We may thus define its resolvent operator for $z \in \mathbb{C}_+ = \{z \in \mathbb{C} : \Im(z) > 0\}$, as

$$G(z) = (A - zI)^{-1}.$$

For $x \in V$, we introduce T_x the subtree rooted at x spanned by the vertices whose common ancestor is x . The trees $T_x, x \in V_o$, are, given N_o , independent with common distribution $\text{GW}(P)$. We denote by A_x the adjacency operator of T_x and set for $z \in \mathbb{C}_+$,

$$G_x(z) = \langle e_x, (A_x - zI)^{-1} e_x \rangle.$$

A well-known consequence of Schur's complement formula is, for all $z \in \mathbb{C}_+$,

$$G_o(z) = - \left(z + \sum_{x \in V_o} G_x(z) \right)^{-1}, \quad (6)$$

see e.g. Klein [33, Proposition 2.1] or [11, 12]. Since G_x and G_o have the same distribution, it follows that G_o satisfies a recursive distribution equation which we are going to study in the regime P close to a Dirac mass and $\Im(z) \rightarrow 0$ in the next subsections.

From (1), G_x is the Cauchy-Stieltjes transform of the random probability measure of \mathbb{R} , $\mu_x = \mu_{T_x}^{e_x}$, i.e.

$$G_x(z) = g_{\mu_x}(z) = \int \frac{1}{\lambda - z} d\mu_x(\lambda).$$

Almost everywhere, the limit

$$G_x(\lambda + i0) := \lim_{\eta \downarrow 0} G_x(\lambda + i\eta)$$

exists and the density of μ_x at $\lambda \in \mathbb{R}$ is given by $\Im G_x(\lambda + i0)/\pi$, see e.g. Simon [40, Chapter 11].

2.2 Skeleton of a Galton-Watson tree

We introduce the subset $S \subset V$ of vertices $x \in V$ such that T_x is an infinite tree. Let π_e be the probability that $o \notin S$, π_e is the extinction probability of T and it is the smallest root of the equation

$$x = \varphi(x),$$

where

$$\varphi(x) = \mathbb{E}(x^{N_o}) = \sum_{k=0}^{\infty} P_k x^k$$

is the moment generating function of P . Also, if $P \neq \delta_1$, the condition $m_1 > 1$ is equivalent to $\pi_e < 1$.

Let N_s and N_e be the number of offspring of the root in S and not in S . The pair (N_s, N_e) has the same distribution than $(\sum_{i=1}^N (1 - \varepsilon_i), \sum_{i=1}^N \varepsilon_i)$ where N has distribution P and is independent of the $(\varepsilon_i)_{i \geq 1}$ an i.i.d. sequence of Bernoulli variables with $\mathbb{P}(\varepsilon_i = 1) = \pi_e = 1 - \mathbb{P}(\varepsilon_i = 0)$. Moreover, conditioned on the root is in S , (N_s, N_e) is conditioned on $N_s \geq 1$. In the sequel (N'_s, N'_e) will denote a pair of random variables with distribution (N_s, N_e) conditioned on $N_s \geq 1$. In particular, the moment generating function of (N'_s, N'_e) is given by

$$\varphi_{s,e}(x, y) = \mathbb{E}\{x^{N'_s} y^{N'_e}\} = \mathbb{E}\{x^{N_s} y^{N_e} | N_s \geq 1\} = \frac{\varphi((1 - \pi_e)x + \pi_e y) - \varphi(\pi_e y)}{1 - \varphi(\pi_e y)}. \quad (7)$$

Similarly, given $o \notin S$, (N_s, N_e) is conditioned on $N_s = 0$. Then, we find easily that the moment generating function of N_e given $o \notin S$ is

$$\varphi_e(x) = \frac{\varphi(\pi_e x)}{\pi_e}. \quad (8)$$

(For more details see Athreya and Ney [7, Section I.12], Durrett [21, Section 2.1]).

We deduce from (6) that the variable G_o^s defined as the law of G_o conditioned on $o \in S$, satisfies the recursive distribution equation,

$$G_o^s(z) \stackrel{d}{=} - \left(z + \sum_{x=1}^{N'_s} G_x^s(z) + V(z) \right)^{-1}, \quad (9)$$

where G_x^s are independent copies of G_o^s , independent of $(N'_s, V(z))$ defined by

$$V(z) = \sum_{x=1}^{N'_e} G_x^e(z), \quad (10)$$

and G_x^e are independent copies of G_o given $o \notin S$ and are independent of (N'_s, N'_e) with moment generating function given by (7).

Now if P is close to δ_q with $q \geq 2$, then the central idea is to interpret (9) has a stochastic perturbation of the deterministic equation, $g(z) \in \mathbb{C}_+$ and

$$g(z) = -(z + qg(z))^{-1}, \quad (11)$$

which characterizes the Cauchy-Stieltjes transform of the semicircle distribution with radius $2\sqrt{q}$. This is the objective of the next subsection. We will use that, as a function of P , the extinction probability is weakly continuous at any $P \neq \delta_1$.

Lemma 10. *The map $P \mapsto \pi_e(P)$ from $\mathcal{P}(\mathbb{Z}_+)$ to $[0, 1]$ is continuous for the weak convergence at any $P \neq \delta_1$*

Proof. Take $P \neq \delta_1$. Fix a sequence of probability measures P_n converging weakly to P . We set $\pi_n = \pi_e(P_n)$, $\pi_\infty = \pi_e(P)$, and we should prove that $\pi_n \rightarrow \pi_\infty$. We denote by φ_n and φ the generating functions of P_n and P . We have the uniform convergence

$$\max_{x \in [0, 1]} |\varphi_n(x) - \varphi(x)| \rightarrow 0, \quad (12)$$

(see Kallenberg [29, Theorem 4.3]). We first prove that $\liminf_n \pi_n \geq \pi_\infty$. Consider a subsequence of $\pi_{n'}$ converging to $\pi' \in [0, 1]$. Using (12) and the continuity of φ , we find that

$$0 = \varphi_{n'}(\pi_{n'}) - \pi_{n'} = \varphi(\pi_{n'}) - \pi_{n'} + o(1) = \varphi(\pi') - \pi' + o(1).$$

In particular $\varphi(\pi') = \pi'$ and $\pi' \geq \pi_\infty$ from the definition of π_∞ .

To conclude of the proof of the lemma, it remains to check that $\limsup_n \pi_n \leq \pi_\infty$. We may assume that $\pi_\infty < 1$ otherwise there is nothing to prove. In particular, since $P \neq \delta_1$, we have $m_1 > 1$ and $P \neq P_0\delta_0 + P_1\delta_1$. Fix any $x \in (\pi_\infty, 1)$, the function φ is strictly convex and $\varphi(x) - x < 0$. From (12), we deduce that for all n large enough, $\varphi_n(x) - x < 0$. Hence, $\pi_n < x$. Since x may be arbitrarily close to π_∞ , we get $\limsup_n \pi_n \leq \pi_\infty$. \square

We will use the straightforward consequence of Lemma 10. Recall that $q \geq 2$.

Corollary 11. *There exists $\varepsilon > 0$ such that if $W_1(P, \delta_q) \leq \varepsilon$ then $\mathbb{E}[N'_s + N'_e] \leq 2q$.*

Proof. By Lemma 10, if ε is small enough, we have $\pi_e(P) \geq 3/4$. Then $\mathbb{E}[N_s + N_e | N_s \geq 1] \leq (4/3)\mathbb{E}[N_s + N_e] \leq 2q$ if ε is small enough. \square

2.3 Convergence of the resolvent in the upper half-plane

We first check that the resolvent converges when $\Im(z) > 0$. The total variation distance between two probability measures P and Q on \mathbb{Z}_+ is classically defined as

$$d_{\text{TV}}(P, Q) = \frac{1}{2} \sum_{k=0}^{\infty} |P(k) - Q(k)|.$$

The total variation distance is a metric for the weak convergence on discrete spaces. We have $d_{\text{TV}}(P, \delta_q) = \mathbb{P}(N \neq q)$ if N has distribution P .

Lemma 12. *For any $z \in \mathbb{C}_+$, if $d_{\text{TV}}(P, \delta_q) \rightarrow 0$ then $G_o(z)$ and $G_o^s(z)$ converge weakly to $g(z)$.*

Proof. Set $\eta = \Im(z) > 0$. In (6), $N = |V_o|$ is independent of G_x , thus we find

$$\begin{aligned} \mathbb{E}|G_o(z) - g(z)| &\leq \eta^{-2} \mathbb{E} \left\{ \left| \sum_{x=1}^q G_x(z) - qg(z) \right| \mathbf{1}_{N=q} \right\} + 2\eta^{-1} \mathbb{P}(N \neq q) \\ &\leq \eta^{-2} q \mathbb{E}|G_o(z) - g(z)| + 2\eta^{-1} d_{\text{TV}}(P, \delta_q). \end{aligned}$$

We find that for all $z \in \mathbb{C}_+$ such that $\Im(z) > \sqrt{q}$, $G_o(z)$ converges in probability to $g(z)$ as $d_{\text{TV}}(P, \delta_q) \rightarrow 0$. Since $z \mapsto G_o(z)$ is bounded by $\Im(z)^{-1}$ and analytic on \mathbb{C}_+ , from Montel's theorem, we may extend this convergence to all z in \mathbb{C}_+ . Finally, by Lemma 10, $\mathbb{P}(o \in S) = 1 - \pi_e$ goes to 1 as $d_{\text{TV}}(P, \delta_q) \rightarrow 0$. Hence, the same result holds for G_o^s . \square

2.4 Tail bounds for the resolvent

The next lemma can be found in [1, Proposition B.2].

Lemma 13. *For any $0 < s < 1$ and $I = [a, b]$, there exists $C = C_s(a, b)$ such that for any probability measure $\mu \in \mathcal{P}(\mathbb{R})$ and $\eta \geq 0$,*

$$\int_I |g_\mu(\lambda + i\eta)|^s d\lambda \leq C.$$

We fix a closed interval $I = [a, b] \subset [-2\sqrt{q}, 2\sqrt{q}]$ and let E be a random variable uniformly sampled on I . It was observed by Aizenman, Sims and Starr [1] that Lemma 13 implies the tightness of the random variables $G_o^s(E + i\eta)$, $G_o^s(E + i0)$ and $V(E + i\eta)$. Observe that $G_o^s(E + i0)$ is well-defined since $G_o(\lambda + i0) = \lim_{\eta \downarrow 0} G_o(\lambda + i\eta)$ exists for almost all $\lambda \in I$ and is measurable as a function of λ . We will use the following corollary of Lemma 13.

Corollary 14. *Let $0 < s < 1$. There exist positive constants C, ε such that, if $W_1(P, \delta_q) \leq \varepsilon$, then for any $\eta \geq 0$ and $t > 0$,*

$$\mathbb{P}(|G_o^s(E + i\eta)| \geq t) \leq Ct^{-s} \quad \text{and} \quad \mathbb{P}(|G_o^s(E + i\eta)|^{-1} \geq t) \leq Ct^{-s}.$$

Proof. The first statement follows directly from Markov inequality and Lemma 13. From the second statement, we may observe that (9) gives that

$$-(G_o^s(z))^{-1} - z = \sum_{x=1}^{N'_s} G_x^s(z) + V(z) = g_\nu(z)$$

is the Cauchy-Stieltjes transform of the finite measure ν on \mathbb{R} whose total mass is equal to $\nu(\mathbb{R}) = N'_s + N'_e$. It follows, by Corollary 11, that $\mathbb{E}\nu(\mathbb{R}) \leq C$ if ε is small enough. However, by Lemma 13 and the linearity of $\mu \mapsto g_\mu$, we get

$$\int_I |(G_o^s(\lambda + i\eta))^{-1} + \lambda + i\eta|^s d\lambda = \int_I |g_\nu(\lambda + i\eta)|^s d\lambda \leq C\nu(\mathbb{R})^s.$$

Taking expectation, we find

$$\mathbb{E}|G_o^s(E + i\eta)^{-1} + E + i\eta|^s \leq C\mathbb{E}\nu(\mathbb{R})^s \leq C_0.$$

To conclude the proof, it remains to use $|x|^s \leq |x + y|^s + |y|^s$ and Markov inequality. \square

Lemma 15. *Let $0 < s < 1$. For any $\eta \geq 0$, $\mathbb{E}|V(E + i\eta)|^s \rightarrow 0$ as $W_1(P, \delta_q) \rightarrow 0$.*

Proof. We use that $V(z) = g_\nu(z)$ where ν is a finite random measure with mass $\nu(\mathbb{R}) = N'_e$. In particular, by Lemma 13, and the linearity of $\mu \mapsto g_\mu$, we find, for some $C > 0$,

$$\mathbb{E}|V(E + i\eta)|^s = \mathbb{E} \frac{1}{\ell(I)} \int_I |g_\nu(\lambda + i\eta)|^s d\lambda \leq C\mathbb{E}(N'_e)^s \leq C\mathbb{E}N'_e.$$

Now, $\mathbb{E}N'_e \leq \mathbb{E}N_e/(1 - \pi_e) = \pi_e \mathbb{E}N/(1 - \pi_e) \leq \pi_e(q + W_1(P, \delta_q))/(1 - \pi_e)$. We conclude with Lemma 10. \square

2.5 Lyapunov exponent

In the spirit of [1], for $z \in \mathbb{C}_+$, we may then define the Lyapunov exponent

$$L_P(z) = -\mathbb{E} \log |G_o^s(z)| - \frac{1}{2} \mathbb{E} \log N'_s.$$

Lemma 16. *The function L_P is a non-negative harmonic function on \mathbb{C}_+ .*

Proof. First, L_P is an harmonic function as it is the real part of the harmonic function $-\mathbb{E} \log G_o^s(z) - \frac{1}{2} \mathbb{E} \log N'_s$. Also, from (9), since $\Im(-1/z) = \Im(z)/|z|^2$, we have

$$\mathbb{E} \log \Im(G_o^s(z)) = \mathbb{E} \log |G_o^s(z)|^2 + \log \Im \left(z + \sum_{x=1}^{N'_s} G_x^s(z) + V(z) \right). \quad (13)$$

Now, $\Im(z) \geq 0$, $\Im V(z) \geq 0$ and $N'_s \geq 1$. Hence, from Jensen inequality,

$$\begin{aligned} \log \Im \left(z + \sum_{x=1}^{N'_s} G_x^s(z) + V(z) \right) &\geq \log \left(\frac{1}{N'_s} \sum_{x=1}^{N'_s} \Im(G_x^s(z)) \right) + \log N'_s \\ &\geq \frac{1}{N'_s} \sum_{x=1}^{N'_s} \log \Im(G_x^s(z)) + \log N'_s. \end{aligned}$$

We now take the expectation and use that G_x^s are independent copies of G_o^s , independent of $N'_s \geq 1$. We obtain that

$$\mathbb{E} \log \Im(G_o^s(z)) \geq 2\mathbb{E} \log |G_o^s(z)| + \mathbb{E} \log \Im(G_o^s(z)) + \mathbb{E} \log N'_s.$$

Hence, $L_P(z) \geq 0$. □

From Corollary 14, if $W_1(P, \delta_q)$ is small enough, for any $\eta \geq 0$, we may define

$$\gamma_P(\eta) = \mathbb{E} L_P(E + i\eta) = -\mathbb{E} \log |G_o^s(E + i\eta)| - \frac{1}{2} \mathbb{E} \log N'_s,$$

where, as above, E is uniform on $I = [a, b]$. If $g \in \mathbb{C}_+$ is defined by (11), it is easy to check that we have for any $\lambda \in [-2, 2]$, $|g(\lambda + i0)|^2 = 1/q$. Hence, for any $\lambda \in [-2, 2]$,

$$L_{\delta_q}(\lambda + i0) = 0 \quad \text{and} \quad \gamma_{\delta_q}(0) = 0. \quad (14)$$

The next key statement, first proved in a similar context in [1], asserts that the averaged Lyapunov exponent is a continuous function of (P, η) .

Proposition 17. *Equip $\mathcal{P}(\mathbb{Z}_+)$ with the W_1 -distance. Then, the function $(P, \eta) \mapsto \gamma_P(\eta)$ is continuous on $\delta_q \times [0, 1]$, that is, for any $0 \leq \eta_0 \leq 1$,*

$$\lim_{W_1(P, \delta_q) \rightarrow 0, \eta \rightarrow \eta_0} \gamma_P(\eta) = \gamma_{\delta_q}(\eta_0).$$

In particular, for any $\varepsilon > 0$, as $W_1(P, \delta_q) \rightarrow 0$,

$$\ell(\lambda \in I : L_P(\lambda + i0) \geq \varepsilon) \rightarrow 0.$$

Proof. The second statement is a direct consequence of the first statement and (14). It is straightforward to adapt the proof of [1, Theorem 3.1] (see there for a more detailed argument). We first bound $L_P(z)$ and write

$$|\log |G_o^s(z)|| = \log_+ |G_o^s(z)| + \log_+ |G_o^s(z)|^{-1},$$

where $\log_+(x) = \log(x \vee 1)$. For the first term, we have the bound $\log_+ |G_o^s(z)| \leq \log_+(\Im(z)^{-1})$. For the second term, from (9),

$$|G_o^s(z)|^{-1} \leq \Im(z) + (N'_s + N_e)\Im(z)^{-1}.$$

In particular, if $z = \lambda + i\eta$, with $\eta \geq 1$,

$$\mathbb{E}|\log |G_o^s(z)|| \leq 2\mathbb{E}\log(\eta + \eta^{-1}(N'_s + N_e)) \leq 2\log(\eta + \eta^{-1}\mathbb{E}(N'_s + N_e)).$$

By Corollary 11, it follows that $L_P(\lambda + i\eta)/\eta \rightarrow 0$ as $\eta \rightarrow \infty$ uniformly for all $\lambda \in \mathbb{R}$ and P such that $W_1(P, \delta_q) \leq \varepsilon$ small enough. We now fix $\eta \geq 0$. Since $z \mapsto L_P(z + i\eta)$ is a non-negative harmonic function on \mathbb{C}_+ , from Nevanlinna's representation theorem, it implies that

$$L_P(z + i\eta) = \Im \int \frac{d\nu_{P,\eta}(\lambda)}{\lambda - z} = \Im(g_{\nu_{P,\eta}}(z)),$$

where $\nu_{P,\eta}$ is a Borel measure on \mathbb{R} such that $\int \frac{d\nu_{P,\eta}(\lambda)}{1+\lambda^2} < \infty$ (see Duren [20]). Since $z \mapsto L_P(z + i\eta)$ has a definite sign, it has locally integrable boundary value [20, Theorem 1.1]. From the inversion formula of Cauchy-Stieltjes transform, we deduce that $\nu_{P,\eta}$ is absolutely continuous with density at λ given by $L_P(\lambda + i\eta)/\pi$ and

$$\gamma_P(\eta) = \frac{1}{\ell(I)} \int_I L_P(\lambda + i\eta) d\lambda = \frac{1}{\pi\ell(I)} \int_I d\nu_{P,\eta}(\lambda).$$

Now, the claimed continuity of $(P, \eta) \mapsto \gamma_P(\eta)$ is a consequence of the vague continuity of $(P, \eta) \mapsto \nu_{P,\eta}$ on $\delta_q \times [0, 1]$. This is in turn a consequence of the continuity for any $z \in \mathbb{C}_+$ of $(P, \eta) \mapsto \Im(g_{\nu_{P,\eta}}(z))$ on $\delta_q \times [0, 1]$ (since the imaginary part of the resolvent characterizes the measure). Now, we recall that $\Im(g_{\nu_{P,\eta}}(z)) = L_P(z + i\eta)$, hence this last continuity follows from (i) $\mathbb{E} \log N'_s \rightarrow \log q$ as $W_1(P, \delta_q) \rightarrow 0$ and (ii) Lemma 12 which implies that for all $z \in \mathbb{C}_+$, $G_o^s(z)$ converges weakly to $g(z)$ when $d_{TV}(P, \delta_q) \rightarrow 0$. \square

2.6 Convergence of the resolvent on the real axis

In this subsection, we will prove the following theorem.

Theorem 18. *Let E be uniform on $I = [a, b]$, as $W_1(P, \delta_q) \rightarrow 0$, $(E, G_o(E + i0))$ converges weakly to $(E, g(E))$. Consequently, for any $\varepsilon > 0$, as $W_1(P, \delta_q) \rightarrow 0$,*

$$\int_I \mathbb{P}(|g(\lambda) - G_o(\lambda + i0)| \geq \varepsilon) d\lambda \rightarrow 0.$$

The main ideas of proof for the above result are again borrowed from [1]. We will use a notion of discrepancy of a non-negative random variable X ,

$$\kappa(X) = \mathbb{E} \left| \frac{X - X'}{X + X'} \right|$$

with the convention that $0/0 = 0$ and X' is an independent copy of the non-negative random variable X . It is easy to check that $\kappa(X) = 0$ is equivalent to X a.s. constant. The next lemma summarizes some properties of κ .

Lemma 19. *Let X, Y be non-negative random variables and $\lambda > 0$. We have $\kappa(X + Y) \leq \kappa(X) + \kappa(Y)$, $\kappa(\lambda X) = \kappa(X)$, $\kappa(1/X) = \kappa(X)$ and,*

$$\kappa(XY) \leq 6\kappa(X) + 6\kappa(Y).$$

Also, if $X = \sum_{i=1}^N Y_i$ with Y_i independent and independent of N , we have for any integer q ,

$$\kappa\left(\sum_{i=1}^N Y_i\right) \leq \mathbb{P}(N \neq q)^2 + \sum_{i=1}^q \kappa(Y_i).$$

Finally, if X_n converges weakly to X then

$$\kappa(X) \leq \liminf_n \kappa(X_n).$$

Proof. Only the last three statement deserves a proof. We write

$$\begin{aligned} \left| \frac{XY - X'Y'}{XY + X'Y'} \right| &\leq \left| \frac{(X - X')Y}{XY + X'Y'} \right| + \left| \frac{(Y - Y')X'}{XY + X'Y'} \right| \\ &\leq (T \vee 1) \left| \frac{X - X'}{X + X'} \right| + (S \vee 1) \left| \frac{Y - Y'}{Y + Y'} \right|, \end{aligned}$$

with $T = Y/Y'$ and $S = X'/X$. Now, if $k = (X' - X)/(X + X')$ and $l = (Y - Y')/(Y + Y')$. We get,

$$T = \frac{1+l}{1-l} \leq \frac{2}{1-l} \quad \text{and} \quad S = \frac{1+k}{1-k} \leq \frac{2}{1-k}.$$

Hence, from Markov inequality, if $t > 2$,

$$\mathbb{P}(T > t) \leq \mathbb{P}(l > 1 - 2/t) \leq \kappa(Y)/(1 - 2/t),$$

and similarly for S . We deduce that for $s, t > 2$,

$$\kappa(XY) \leq t\kappa(X) + s\kappa(Y) + \kappa(Y)/(1 - 2/t) + \kappa(X)/(1 - 2/s).$$

We finally choose $s = t = 3$ and get the required bound on $\kappa(XY)$.

If $X = \sum_{i=1}^N Y_i$, we use that

$$\left| \frac{\sum_{i=1}^N Y_i - \sum_{i=1}^{N'} Y'_i}{\sum_{i=1}^N Y_i + \sum_{i=1}^{N'} Y'_i} \right| \leq \mathbf{1}((N, N') \neq (q, q)) + \left| \frac{\sum_{i=1}^q Y_i - \sum_{i=1}^q Y'_i}{\sum_{i=1}^q Y_i + \sum_{i=1}^q Y'_i} \right|.$$

Finally, the statement about $\kappa(X_n)$ is a direct consequence of Fatou's lemma. □

Lemma 20. *There exists $\varepsilon > 0$ such that if $W_1(P, \delta_q) \leq \varepsilon$ then, for any $z \in \mathbb{C}_+$,*

$$\kappa(\Im G_o^s(z)) \leq \sqrt{2L_P(z)},$$

and

$$\kappa(|G_o^s(z)|^2) \leq 6\text{d}_{\text{TV}}(P, \delta_q)^2 + 6\sqrt{2}(q+1)\sqrt{L_P(z)} + 6\mathbb{P}(N'_e \geq 1).$$

Proof. We use the following second order refinement of Jensen inequality, for any integer $n \geq 2$ and positive x_i ,

$$\log \left(\frac{1}{n} \sum_{i=1}^n x_i \right) \geq \frac{1}{n} \sum_{i=1}^n \log x_i + \frac{1}{2n(n-1)} \sum_{i \neq j} \left(\frac{x_i - x_j}{x_i + x_j} \right)^2,$$

(proved in [1, Lemma 4.1]). Let $z \in \mathbb{C}_+$. From (13), we find

$$\begin{aligned} \log \Im(G_o^s(z)) &\geq \log |G_o^s(z)|^2 + \frac{1}{N'_s} \sum_{x=1}^{N'_s} \log \Im(G_x^s(z)) \\ &\quad + \frac{1}{2N'_s(N'_s-1)} \sum_{x \neq y} \left(\frac{\Im(G_x^s(z)) - \Im(G_y^s(z))}{\Im(G_x^s(z)) + \Im(G_y^s(z))} \right)^2 + \log N'_s, \end{aligned}$$

Since N'_s is independent of G_x^s , taking expectation, we get that

$$\mathbb{E} \frac{1}{N'_s(N'_s-1)} \sum_{x \neq y} \left(\frac{\Im(G_x^s(z)) - \Im(G_y^s(z))}{\Im(G_x^s(z)) + \Im(G_y^s(z))} \right)^2 \leq L_P(z).$$

By Lemma 10, if $W_1(P, \delta_q) \leq \varepsilon$ is small enough, then $\mathbb{P}(N'_s \geq 2) \geq 1/2$, we deduce that

$$\kappa^2(\Im G_o^s(z)) \leq \mathbb{E} \left(\frac{\Im(G_1^s(z)) - \Im(G_2^s(z))}{\Im(G_1^s(z)) + \Im(G_2^s(z))} \right)^2 \leq 2L_P(z),$$

where $(G_i^s(z))$, $i = 1, 2$ are independent copies of G_o^s .

Similarly, from (9)

$$\Im(G_o^s(z)) = |G_o^s(z)|^2 \Im \left(z + \sum_{x=1}^{N'_s} \Im(G_x^s(z)) + \Im V(z) \right).$$

We obtain from Lemma 19 that

$$\kappa(|G_o^s(z)|^2) \leq 6\kappa(\Im G_o^s(z)) + 6\kappa \left(\sum_{x=1}^{N'_s} \Im(G_x^s(z)) \right) + 6\kappa(\Im V(z)).$$

We find from another use of Lemma 19,

$$\kappa(|G_o^s(z)|^2) \leq 6\text{d}_{\text{TV}}(P, \delta_q)^2 + 6(q+1)\kappa(\Im G_o^s(z)) + 6\kappa(\Im V(z)).$$

Finally, $\kappa(\Im V(z)) \leq \mathbb{P}(V(z) \neq 0) \leq \mathbb{P}(N'_e \geq 1)$. It concludes the proof. \square

We are now ready to prove Theorem 18.

Proof of Theorem 18. First, by Lemma 10, $\mathbb{P}(o \in S) \rightarrow 1$ as $d_{TV}(P, \delta_q) \rightarrow 0$. It is thus sufficient to prove the result with the conditioned variable G_o^s instead of G_o . The proof is then essentially contained in [1, Section 5]. Let us briefly sketch their argument. From Corollary 14, the pair of random variables $(E, G_o^s(E))$ is tight as $W_1(P, \delta_q) \rightarrow 0$. Let us consider an accumulation point (E, Z) . From Corollary 14, $1/Z$ is a proper random variable on \mathbb{C} . We denote by P_λ the conditional distribution of Z given $E = \lambda$ (it is defined for almost all $\lambda \in I$ from Fubini's Theorem). From the continuous mapping theorem, Lemma 15 and (9),

$$-Z^{-1} \stackrel{d}{=} E + \sum_{i=1}^q Z_i,$$

where, given E , Z_i are independent copies of Z . Notably, if $Z(\lambda)$, $Z_i(\lambda)$ are independent with distribution P_λ , for almost all $\lambda \in I$,

$$-Z^{-1}(\lambda) \stackrel{d}{=} \lambda + \sum_{i=1}^q Z_i(\lambda). \quad (15)$$

However, from Lemma 20, Proposition 17 and Fubini's Theorem,

$$\int_I \mathbb{E} \kappa(\Im(Z(\lambda))) d\lambda = \int_I \mathbb{E} \kappa(|Z(\lambda)|^2) d\lambda = 0.$$

We deduce that for almost all $\lambda \in I$, $\Im(Z(\lambda))$ and $|Z(\lambda)|$ are supported on a single point. In particular, P_λ is supported on at most 2 points. Assume that $Z(\lambda)$ can take two distinct values with positive probability, say (z_1, z_2) . From (15), since $q \geq 2$, $-Z^{-1}(\lambda)$ could take at least three different values with positive probability: $(\lambda + qz_1, \lambda + qz_2, \lambda + (q-1)z_1 + z_2)$. It contradicts the fact that the support of P_λ has at most two points.

Finally, if, for almost all $\lambda \in I$, $P_\lambda = \delta_{z(\lambda)}$ then, from (15), $-z(\lambda)^{-1} = \lambda + qz(\lambda)$. Since $\Im(z(\lambda)) \geq 0$ it implies that $z(\lambda) = g(\lambda)$. \square

2.7 Proof of Theorem 1, Corollary 2 and Theorem 3

We start by recalling the probabilistic version of Scheffé's Lemma.

Lemma 21 (Scheffé's Lemma). *Let X_n be a sequence of non-negative random variables converging in probability to $X \in L^1(\mathbb{P})$. Then $\mathbb{E}X_n \rightarrow \mathbb{E}X$ implies $\mathbb{E}|X_n - X| \rightarrow 0$.*

Proof. By dominated convergence, $\mathbb{E}(X_n \wedge X) \rightarrow \mathbb{E}X$. However, $|X_n - X| = X + X_n - 2(X_n \wedge X)$. \square

Proof of Theorem 1. Let f be the density of the absolutely continuous part of the random measure $\mu_T^{e_o}$. We have for almost all λ , $f(\lambda) = \Im G_o(\lambda + i0)/\pi$. From Theorem 18 applied to $I = [-2\sqrt{q}, 2\sqrt{q}]$, for any $\varepsilon > 0$, if $W_1(P, \delta_q)$ is small enough,

$$\mathbb{E} \int f(\lambda) d\lambda \geq \int_{-2\sqrt{q}}^{2\sqrt{q}} f_q(\lambda) d\lambda - \varepsilon = 1 - \varepsilon.$$

It follows that

$$\lim_{P \xrightarrow{L_1} \delta_q} \mathbb{E} \int f(\lambda) d\lambda = 1.$$

It remains to use Theorem 18 with Scheffé's Lemma 21 for the probability measure $\mathbb{P} \otimes U$ where U is the uniform probability measure on $[-2\sqrt{q}, 2\sqrt{q}]$. \square

Proof of Corollary 2. By definition, $\bar{f}(\lambda) \geq \mathbb{E}f(\lambda)$. Hence from Theorem 1, for any Borel I ,

$$\lim_{P \xrightarrow{L_1} \delta_q} \int_I \bar{f}(\lambda) d\lambda \geq \int_I f_q(\lambda) d\lambda.$$

Applied to $I = [-2\sqrt{q}, 2\sqrt{q}]$, we deduce that the above inequality is an equality. That is, for any Borel I ,

$$\lim_{P \xrightarrow{L_1} \delta_q} \int_I \bar{f}(\lambda) d\lambda = \int_I f_q(\lambda) d\lambda.$$

We conclude with a new use of Scheffé's Lemma 21. \square

Proof of Theorem 3. We start by observing that $W_2(P, \delta_{q+1}) \rightarrow 0$ implies that $W_1(\hat{P}, \delta_q) \rightarrow 0$. Also, from Schur's formula (6)

$$G_o(z) \stackrel{d}{=} - \left(z + \sum_{i=1}^N \hat{G}_i(z) \right)^{-1},$$

where N has distribution P , independent of $(\hat{G}_i(z))_{i \geq 1}$, independent copies of $\hat{G}_o(z)$, the resolvent at the root of the adjacency operator of a $\text{GW}(\hat{P})$ tree.

Now, the Cauchy-Stieltjes transform of \check{f}_q is \check{g} which satisfies the identity,

$$\check{g}(z) = -(z + (q+1)g(z))^{-1},$$

where $g(z)$ denotes the Cauchy-Stieltjes transform of f_q (see [11, Eqn (5)]). It remains to apply Theorem 18 and the continuous mapping theorem. We find that, if E is uniform on $I = [a, b]$, then $(E, G_o(E + i0))$ converges weakly to $(E, \check{g}(E))$. We may then repeat the argument of Theorem 1 and Corollary 2. \square

2.8 Other approaches to the existence of continuous spectrum

In the above argument, we have followed the strategy of [1] to prove the existence of continuous spectrum for random Schrödinger operators on infinite trees at small disorder. Other approaches of this result have been proposed, they all start from the analog of the recursive distribution equation coming from Schur's formula (6). The original proof of Klein [33] relies on an application of the implicit function theorem, see also [35, 34], a more geometric study of the fixed point equation was initiated by Froese, Hasler and Spitzer [24] and further developed in [26, 25, 28, 31, 30].

A common point of all these methods is that they require more moment conditions on the disorder than the one obtained in Lemma 15. In fact, if we restrict our attention to a specific Borel subsets we can improve drastically on Lemma 15. This is the content of the next statement.

Proposition 22. *Let $\varepsilon > 0$ and $p \geq 1$. There exists a Borel set $K \subset \mathbb{R}$ (depending on ε and p) such that $\ell(K^c) \leq \varepsilon$ and, as $W_p(P, \delta_q) \rightarrow 0$,*

$$\sup \{ \mathbb{E} |V(\lambda + i\eta)|^p : \eta \geq 0, \lambda \in K \} \rightarrow 0.$$

Before proving this proposition, let us simply mention that it could be used to give alternative proofs of Theorem 1 and Theorem 3 following the approach of Keller et al. [31, 30]. The approach of Klein does not seem however to accommodate easily with vertices with only one offspring in the skeleton tree. We will however not pursue further in this direction here and restrict ourselves to the proof of Proposition 22.

We start with a standard lemma on the total progeny of subcritical Galton-Watson trees.

Lemma 23 (Total progeny of subcritical Galton-Watson tree). *Let Q be a probability measure on non-negative integers whose moment generating function ψ satisfies $\psi(\rho) < \rho$ for some $\rho > 1$. Let Z be the total number of vertices in a GW(Q) tree. We have for any $t \geq 1$,*

$$\mathbb{P}(Z \geq t) \leq \rho \left(\frac{\psi(\rho)}{\rho} \right)^t.$$

Proof. The proof is extracted from [21, Theorem 2.3.1]. Let $Y_i, i \geq 1$, be iid copies with distribution Q and $X_i = Y_i - 1$. Consider the random walk, $S_0 = 1$ and for $t \geq 1$, $S_t = S_0 + \sum_{i=1}^t X_i$. The total progeny Z has the same distribution that the hitting time $\tau = \inf\{t \geq 1 : S_t = 0\}$ (see for example [21, Section 2.1]). Set $\theta = \log \rho$, $f(\theta) = \mathbb{E} e^{\theta X} = \psi(e^\theta) e^{-\theta} = \psi(\rho)/\rho < 1$. By construction $M_t = e^{\theta S_t} / f(\theta)^t$ is non-negative martingale with mean $M_0 = e^\theta = \rho$ with respect to the filtration $\mathcal{F}_t = \sigma(S_0, \dots, S_t)$. From Doob's optional stopping time theorem, we have

$$\mathbb{E}[M_\tau] = \mathbb{E}[f(\theta)^{-\tau}] = \rho.$$

Then, since $0 < f < 1$, from Markov inequality,

$$\mathbb{P}(\tau \geq t) = \mathbb{P}(f(\theta)^{-\tau} \geq f(\theta)^{-t}) \leq \rho f(\theta)^t.$$

Since $\mathbb{P}(Z \geq t) = \mathbb{P}(\tau \geq t)$ it concludes the proof. \square

Proof of Proposition 22. We first observe that for any $0 < \delta < 1$ and all $0 \leq y \leq y_\delta = (\delta/2)^{1/(q-1)}$, we have $y^q \leq (\delta/2)y$. The parameter $\delta > 0$ will be fixed later on in the proof. We also fix some $0 < y < y_\delta$.

As usual, let φ be the moment generating function of P . Now, for $x \in [0, 1]$, since for any $a, b \geq 0$, $|x^a - x^b| \leq |a - b|$, we find that $|\varphi(x) - x^q| \leq \mathbb{E}|N - q| = W_1(P, \delta_q)$. From what precedes for all P such that $W_1(P, \delta_q) \leq \delta y/2$, we find that $\varphi(y) \leq \delta y$.

From [7, Section 12, Theorem 3], $G_o^e(z)$ is the resolvent at the root of the adjacency matrix of a random tree T , a subcritical GW(Q) tree where Q has moment generating function φ_e is given by (8). From what precedes, we deduce that $\varphi_e(\rho) \leq \delta\rho$, with $\rho = y/\pi_e$. By Lemma 10, if $W_1(P, \delta_q)$ is small enough then $\rho > 1$. Hence, from Lemma 23, if $|T|$ is the total number of vertices in T , we deduce that for all $k \geq 1$,

$$\mathbb{P}(|T| \geq k) \leq \frac{y}{\pi_e} \delta^k. \quad (16)$$

as soon as $W_1(P, \delta_q)$ is small enough.

Now, for integer $k \geq 1$, let Λ_k be the set of real numbers λ such that there exists a tree with k vertices and λ is an eigenvalue of this tree. Obviously, $|\Lambda_k|$ is bounded by k times the number of unlabeled trees with k vertices. In particular, for some $c > 1$,

$$|\Lambda_k| \leq c^k, \quad (17)$$

see Flajolet and Sedgewick [23, Section VII.5]. We define $B_{k,\varepsilon} = \{x \in \mathbb{R} : \exists \lambda \in \Lambda_k, |\lambda - x| \leq \varepsilon 2^{-k}/|\Lambda_k|\}$ and $K = \mathbb{R} \setminus \cup_{k \geq 1} B_{k,\varepsilon}$. By construction,

$$\ell(K^c) \leq \sum_{k \geq 1} |\Lambda_k| \frac{\varepsilon 2^{-k}}{|\Lambda_k|} = \varepsilon.$$

Also, we have for any probability measure ν , $|g_\nu(z)| \leq 1/d(z, \text{supp}(\nu))$. Hence, we find from (16)-(17), for any $\lambda \in K$,

$$\begin{aligned} \mathbb{E}|G_o^e(\lambda + i\eta)|^p &= \sum_{k=1}^{\infty} \mathbb{P}(|T| = k) \mathbb{E}[|G_o^e(\lambda + i\eta)|^p | |T| = k] \\ &\leq \sum_{k=1}^{\infty} \frac{y}{\pi_e} \delta^k \left(2^k |\Lambda_k| / \varepsilon\right)^p \\ &\leq \frac{y}{\pi_e \varepsilon^p} \sum_{k=1}^{\infty} (\delta(2c)^p)^k \end{aligned}$$

Hence, if δ was chosen such that $\delta(2c)^p \leq 1/2$, we obtain,

$$\mathbb{E}|G_o^e(\lambda + i\eta)|^p \leq \frac{y}{\pi_e \varepsilon^p}.$$

Finally, from the definition of $V(z)$ in (10), we have, using Hölder inequality,

$$\begin{aligned} \mathbb{E}|V(\lambda + i\eta)|^p &= \mathbb{E} \left| \sum_{x=1}^{N'_e} G_x^e(\lambda + i\eta) \right|^p \\ &\leq \mathbb{E} \left\{ (N'_e)^{p-1} \sum_{x=1}^{N'_e} |G_x^e(\lambda + i\eta)|^p \right\} \\ &= \mathbb{E} \{(N'_e)^p\} \mathbb{E}|G_o^e(\lambda + i\eta)|^p. \end{aligned}$$

Now, $\mathbb{E}(N'_e)^p \leq \mathbb{E}N_e^p/(1-\pi_e)$ and, as already pointed, $N_e \stackrel{d}{=} \sum_{i=1}^N \varepsilon_i$ where N has distribution P , independent of $(\varepsilon_i)_{i \geq 1}$ an i.i.d. sequence of Bernoulli variables with $\mathbb{P}(\varepsilon_i = 1) = \pi_e = 1 - \mathbb{P}(\varepsilon_i = 0)$. In particular, Hölder inequality implies that

$$\mathbb{E}N_e^p \leq \mathbb{E}N^p \mathbb{E}\varepsilon^p = \mathbb{E}N^p \pi_e.$$

We have thus proved that

$$\mathbb{E}|V(\lambda + i\eta)|^p \leq \frac{y\mathbb{E}N^p}{(1-\pi_e)\varepsilon^p}.$$

Since y can be taken arbitrarily small as $W_1(P, \delta_q) \rightarrow 0$, the conclusion follows. \square

3 Deterministic resolvent bounds

In this section, we state some general relations involving resolvent matrix and delocalization.

3.1 Convergence and matching moments

The objective of this subsection is to compare the Stieltjes transforms of two measures whose first moments coincide.

Proposition 24. *Let μ_1, μ_2 be two real probability measures such that for all integers $1 \leq k \leq n$,*

$$\int \lambda^k d\mu_1(\lambda) = \int \lambda^k d\mu_2(\lambda).$$

Let $\zeta = e^2\pi$. If μ_1 and μ_2 have support in $[-b, b]$ then for all $z \in \mathbb{C}_+$ with $\Im(z) \geq \zeta b \lceil \log n \rceil / n$,

$$|g_{\mu_1}(z) - g_{\mu_2}(z)| \leq \frac{2}{\zeta n b}.$$

Proof. We set

$$g_z(\lambda) = \frac{1}{\lambda - z}.$$

For integer $k \geq 0$, we have

$$\|\partial^{(k)} g_z\|_\infty = k! \eta^{-k-1}.$$

From Jackson's theorem [18, Chap. 7, §8], there exists a polynomial p_z of degree n such that for any $\lambda \in [-b, b]$ and $k \leq n$,

$$|g_z(\lambda) - p_z(\lambda)| \leq \left(\frac{\pi}{2}b\right)^k \frac{(n-k+1)!}{(n+1)!} \|\partial^{(k)} g_z\|_\infty. \quad (18)$$

We take $k = \lceil \log n \rceil$ and $\eta \geq \zeta b \lceil \log n \rceil / n$. Using, $k! \leq k^k$, $\log n / n \leq e^{-1}$, we get,

$$|g_z(\lambda) - p_z(\lambda)| \leq \frac{1}{\eta} \left(\frac{\pi b k}{2\eta(n+2-k)} \right)^k \leq \frac{1}{\eta} \left(\frac{1}{2e^2} \frac{1}{1-e^{-1}} \right)^k \leq \frac{1}{\eta n^2} \leq \frac{1}{\zeta b n}.$$

The conclusion follows. \square

As an immediate corollary, we have the following statement.

Corollary 25. *For $i = 1, 2$, let (G_i, o) be a rooted graph and denote by A_i their adjacency operators. Assume further that the rooted subgraphs $(G_1, o)_h$ and $(G_2, o)_h$ are isomorphic. If, for $i = 1, 2$, $\|A_i\| \leq b$ then for all $z \in \mathbb{C}_+$ with $\Im(z) \geq \zeta b \lceil \log 2h \rceil / (2h)$,*

$$|\langle e_o, (A_1 - z)^{-1} e_o \rangle - \langle e_o, (A_2 - z)^{-1} e_o \rangle| \leq \frac{1}{\zeta b h}.$$

Proof. By assumption and (1), we can apply Proposition 24 to $n = 2h$. \square

3.2 Regularity and resolvent

In this paragraph, for any interval I , we state a weak bound of $\mu(I)$ in terms of g_μ . Much stronger statements have appeared in the literature, see for example [22, 41, 10]. There are however not really adapted to quantum percolation due to the typical presence of a dense atomic part. The next statement gives a weak regularity result.

Lemma 26. *Let μ be a probability measure on \mathbb{R} such that for some $\lambda \in \mathbb{R}$ and $a, b, \eta > 0$,*

$$\Im(g_\mu(\lambda + i\eta)) \geq a \quad \text{and for all } y \geq \eta, \quad \Im(g_\mu(\lambda + iy)) \leq b.$$

Then, if $I = [\lambda - s/2, \lambda + s/2]$ or $I = (\lambda - s/2, \lambda + s/2)$ with $\ell(I) = s \geq 2\eta$, we have

$$\frac{a}{2\rho} \leq \frac{\mu(I)}{\ell(I)} \leq b,$$

where the left-hand side inequality holds if $\rho = s/\eta \geq 8b/a$.

Proof. We have the bound,

$$\Im g_\mu(x + iy) = \int \frac{y}{(x - \lambda)^2 + y^2} d\mu(\lambda) \geq \frac{\mu([x - y, x + y])}{2y}. \quad (19)$$

Applied to $x = \lambda$ and $y = s/2$, it readily implies the upper bound of the lemma.

For the lower bound, let $I_0 = I = (x - t\eta, x + t\eta)$ and for $k \geq 1$, $I_k = [x - t2^k\eta, x - t2^k\eta] \setminus I_{k-1}$. We write

$$\Im g_\mu(x + i\eta) \leq \frac{\mu(I)}{\eta} + \sum_{k=1}^{\infty} \frac{\mu(I_k)}{\eta(1 + 4^k t^2)} \leq \frac{\mu(I)}{\eta} + \sum_{k=1}^{\infty} \frac{\mu(I_k)}{\eta 4^k t^2}$$

From (19), for $x = \lambda$ and $y = t2^k\eta$, $\mu(I_k) \leq \mu([x - t2^k\eta, x - t2^k\eta]) \leq 2^{k+1}t\eta b$, and

$$\frac{\mu(I)}{\eta} \geq a - \sum_{k=1}^{\infty} \frac{2b}{2^k t} = a - \frac{2b}{t}.$$

Setting $2t\eta = s = \ell(I)$, we deduce that

$$\rho \frac{\mu(I)}{\ell(I)} \geq a - \frac{4b}{\rho} \geq \frac{a}{2},$$

if $\rho \geq 8b/a$. \square

Let G be a finite graph with n vertices, A its adjacency matrix, and $\lambda_1, \dots, \lambda_n$ its eigenvalues. For $I \subset \mathbb{R}$, we denote by

$$\Lambda_I = \{k : \lambda_k \in I\}.$$

By definition, we have $\mu_G(I) = n|\Lambda_I|$. In the sequel, ψ_1, \dots, ψ_n is a orthonormal basis of eigenvectors of A , $A\psi_k = \lambda_k\psi_k$. Finally, the resolvent of A is denoted by $R(z) = (A - zI)^{-1}$.

Corollary 27. *Let G be as above and let $o \in V(G)$. Assume that for some $\lambda \in \mathbb{R}$ and $a, b, \eta > 0$,*

$$\Im(R_{oo}(\lambda + i\eta)) \geq a \quad \text{and for all } y \geq \eta, \quad \Im(R_{oo}(\lambda + iy)) \leq b.$$

Then, if $I = [\lambda - s/2, \lambda + s/2]$ or $I = (\lambda - s/2, \lambda + s/2)$ with $\ell(I) = s \geq 2\eta$, we have

$$\frac{as}{2\rho} \leq \sum_{k \in \Lambda_I} |\psi_k(o)|^2 \leq bs,$$

where the left-hand side inequality holds if $\rho = s/\eta \geq 8b/a$.

Proof. We recall that $R_{oo}(z) = g_{\mu_G^o}(z)$ and $\mu_G^o(I) = \sum_{k \in \Lambda_I} |\psi_k(o)|^2$. It thus remains to apply Lemma 26. \square

The following elementary lemma will also be useful.

Lemma 28. *Let U be an open set, $t > 0$ and μ a probability measure on \mathbb{R} , then*

$$\mu(U) \leq 2 \int_U \Im g_\mu(x + it) dx$$

Proof. From (19), it is sufficient to prove that

$$\mu(U) \leq \frac{1}{t} \int_U \mu([x - t, x + t]) dx.$$

Assume first that $U = (a, b)$ is an open interval and that μ is absolutely continuous with density f . Then, we write

$$\int_U \mu([x - t, x + t]) dx = \int_a^b \int_{x-t}^{x+t} f(y) dy dx = \int_{a-t}^{b+t} f(y) \int_{y-t}^{y+t} \mathbf{1}(x \in (a, b)) dx dy.$$

For any $y \in (a, b)$, $\int_{y-t}^{y+t} \mathbf{1}(x \in (a, b)) dx \geq t$. We deduce that

$$\int_U \mu([x - t, x + t]) dx \geq t \int_a^b f(y) dy = t\mu(U).$$

Since, any open set in \mathbb{R} is a countable union of disjoint intervals. We obtain by linearity the claimed result when μ is absolutely continuous. In the general case, we consider a sequence of absolutely continuous probability measures μ_n which converge weakly to μ . Since U is open and $[x - t, x + t]$ is closed, $\mu(U) \leq \liminf \mu_n(U)$ and $\mu([x - t, x + t]) \geq \limsup \mu_n([x - t, x + t])$. We may thus take the limit in the inequality for μ_n . \square

4 Rates of convergence in percolation graphs

In this section, we prove Theorem 5.

4.1 Concentration Lemma

We start by recalling a useful concentration lemma in the context of percolation. Recall that the total variation norm of $f : \mathbb{R} \rightarrow \mathbb{R}$ is

$$\|f\|_{\text{TV}} := \sup \sum_{k \in \mathbb{Z}} |f(x_{k+1}) - f(x_k)|,$$

where the supremum runs over all sequences $(x_k)_{k \in \mathbb{Z}}$ such that $x_{k+1} \geq x_k$ for any $k \in \mathbb{Z}$. If $f = \mathbf{1}_{(-\infty, s]}$ for some real s then $\|f\|_{\text{TV}} = 1$, while if f has a derivative in $L^1(\mathbb{R})$, we get $\|f\|_{\text{TV}} = \int |f'(t)| dt$. The following lemma is a consequence of [9, Lemma C.2].

Lemma 29. *Let $p \in [0, 1]$ and $H = \text{perc}(G, p)$ where G is a finite deterministic graph. Then, for any $f : \mathbb{R} \rightarrow \mathbb{C}$ such that $\|f\|_{\text{TV}} \leq 1$ and every $t \geq 0$,*

$$\mathbb{P} \left(\left| \int f d\mu_H - \mathbb{E} \int f d\mu_H \right| \geq t \right) \leq 2 \exp \left(-\frac{nt^2}{8} \right).$$

4.2 Proof of Theorem 5

We have

$$\mathbb{E} \int \varphi d\mu_{\text{perc}(G, p)} - \int \varphi d\mu = \frac{1}{n} \sum_{v=1}^n \left(\mathbb{E} \int \varphi d\mu_{\text{perc}(G, p)}^{e_v} - \mathbb{E} \int \varphi d\mu_{\text{perc}(\Gamma, p)}^{e_o} \right).$$

Now, if $v \in B(h)$, then

$$\left| \mathbb{E} \int \varphi d\mu_{\text{perc}(G, p)}^{e_v} - \mathbb{E} \int \varphi d\mu_{\text{perc}(\Gamma, p)}^{e_o} \right| \leq 2\|\varphi\|_{\infty}.$$

Otherwise, if $v \notin B(h)$, then $\mathbb{E} \int x^k d\mu_{\text{perc}(G, p)}^{e_v} = \mathbb{E} \int x^k d\mu_{\text{perc}(\Gamma, p)}^{e_o}$ for all $k \leq 2h$. Hence, Jackson's Theorem (see (18)) implies that

$$\left| \mathbb{E} \int \varphi d\mu_{\text{perc}(G, p)}^{e_v} - \mathbb{E} \int \varphi d\mu_{\text{perc}(\Gamma, p)}^{e_o} \right| \leq 2 \left(\frac{\pi}{2} d \right)^k \frac{(2h - k + 1)!}{(2h + 1)!} \|\partial^{(k)} \varphi\|_{\infty}.$$

To conclude the proof of Theorem 5, it remains to use Lemma 29.

4.3 Proof of Corollary 6

Let $\tilde{\mu} = \mu_{\text{perc}(G, p)}$. Using Corollary 25 in the proof of Theorem 5, we find for $\Im(z) \geq \eta_1 = \zeta d [\log 2h] / (2h)$,

$$|g_{\tilde{\mu}}(z) - g_{\mu}(z)| \leq t + \frac{4}{\zeta d \log 2h} \frac{h B_{\Gamma}(h)}{n} + \frac{1}{\zeta d h} \leq t + \frac{5\delta}{\zeta d}.$$

with probability at least $1 - 2 \exp(-nt^2/(8(\pi/\eta_1)^2))$. For $t = 18\delta/(\zeta d)$, we deduce that $|g_{\tilde{\mu}}(z) - g_{\mu}(z)| \leq 23\delta/(\zeta d)$ with probability at least $1 - 2 \exp(-cn\delta^2/h^2)$ with $c = 18^2/(32\pi^2) \geq 1$. It remains to use the numerical value of $\zeta \in (23, 24)$.

5 Weak delocalization in percolation graphs

In this section, we prove Theorem 9 and Theorem 7.

5.1 Proof of Theorem 7

Let $S = [-d, d]$, f be the density of the absolutely continuous part of μ and $h_{\eta}(\lambda) = \frac{1}{\pi} \Im g_{\mu}(z)$ with $z = \lambda + i\eta$. Recall that a.e. $f(\lambda) = \lim_{\eta \rightarrow 0} h_{\eta}(\lambda)$. By assumption,

$$m = \int_S f(\lambda) d\lambda = \mu_{ac}(\mathbb{R}),$$

is positive. By monotone convergence, for any $0 < \varepsilon < m$, we can find a pair (a, b) of positive numbers satisfying,

$$\int_S f(\lambda) \mathbf{1}(a \leq f(\lambda) \leq b) d\lambda \geq m - \varepsilon/2.$$

Also, if $a' = a/2$, $b' = 2b$, and $\eta_0 > 0$, we define the Borel set

$$K_0 = \{\lambda \in S : h_{\eta}(\lambda) \in [a', b'] \text{ for all } \eta \in [0, \eta_0]\}.$$

From Egorov's Theorem, if η_0 is small enough, for any $0 \leq \eta \leq \eta_0$,

$$\int_{K_0} h_{\eta}(\lambda) d\lambda \geq m - \varepsilon. \quad (20)$$

Using the bounds, for all $\lambda \in S$ and $\eta_0 \leq \eta \leq 1$, $\eta_0/(4d^2 + \eta_0^2) \leq h_{\eta}(\lambda) \leq 1/\eta_0$, we find that for some positive a'', b'' ,

$$K_0 \subset K_1 = \{\lambda \in S : h_{\eta}(\lambda) \in [a'', b''] \text{ for all } \eta \in [0, 1]\}.$$

We now set $\tilde{\mu} = \mu_{\text{perc}(G, p)}$ and $\tilde{h}_{\eta}(\lambda) = \frac{1}{\pi} \Im g_{\tilde{\mu}}(z)$, $z = \lambda + i\eta$. By Corollary 6, for any $\eta \geq \eta_1 = 20d \log(2h)/h$ and $\lambda \in \mathbb{R}$, we have

$$|\tilde{h}_{\eta}(\lambda) - h_{\eta}(\lambda)| \leq \delta/\pi$$

with probability at least $1 - 2 \exp(-n\delta^2/h^2)$. Recall that $|g_{\mu}(z) - g_{\mu}(z')| \leq |z - z'|/(\Im(z) \wedge \Im(z'))^2$. Consider a $\delta\eta_1^2$ -net of $(4d + 2)/(\delta\eta_1^2) \leq h^2/\delta$ points on the boundary of the rectangle $R = \{z : \eta_1 \leq \Im(z) \leq 1, \Re(z) \in S\}$. From the maximum principle and the union bound, we deduce easily that

$$\sup_{\lambda + i\eta \in R} |\tilde{h}_{\eta}(\lambda) - h_{\eta}(\lambda)| \leq \frac{2\delta}{\pi} \leq \delta. \quad (21)$$

with probability at least $1 - \delta^{-1}h^2 \exp(-n\delta^2/h^2)$.

Up to modifying the constants c_0, c_1 in the statement of the theorem, we can assume without loss of generality that $\eta_1 \leq \eta_0$. Hence, on the event (21), for all $\lambda \in K_1$, $\eta_1 \leq \eta \leq 1$, $\tilde{h}_\eta(\lambda) \in [a'' - \delta, b'' + \delta]$. It remains to apply Lemma 26 to all $\lambda \in K_1$ and set $K = \bar{K}_1$. We deduce the first statement of Theorem 7 by adjusting all constants.

Also, since $\int \tilde{h}_\eta(\lambda) d\lambda = 1$, we find from (20) that, on the event (21),

$$\int_{K^c} \tilde{h}_\eta(\lambda) d\lambda \leq 1 - m + \varepsilon + \delta.$$

Applying Lemma 28, we obtain the second statement of Theorem 7.

5.2 Concentration lemma for local graph functionals

To prove Theorem 9, we need a basic concentration lemma for local functions of the graph. To this end, we denote by \mathcal{G}^* the set of finite rooted graphs, i.e. the set of pairs (G, o) formed by a finite graph $G = (V, E)$ and a distinguished vertex $o \in V$. Recall that for integer $h \geq 1$, we denote by $(G, o)_h$ the subgraph of G spanned by the vertices which are at distance at most h from o . We shall say that a function τ from \mathcal{G}^* to \mathbb{R} is h -local, if $\tau(G, o)$ is only function of $(G, o)_h$.

The next statement is a straightforward corollary of Azuma-Hoeffding's inequality. Recall that the parameter $M_h(G)$ was defined in (5).

Lemma 30. *Let $p \in [0, 1]$ and $H = \text{perc}(G, p)$ where G is a deterministic graph on n vertices. If $\tau : \mathcal{G}^* \rightarrow [0, 1]$ is h -local then for any $t \geq 0$,*

$$\mathbb{P} \left(\sum_{v \in V(G)} \tau(H, v) - \mathbb{E} \sum_{v \in V(G)} \tau(H, v) \geq nt \right) \leq \exp \left(-\frac{nt^2}{2M_h^2(G)} \right).$$

Proof. We may assume that the vertex V of G is $\{1, \dots, n\}$. Let A be the adjacency matrix of H . For $2 \leq k \leq n$, we define the vector $X_k = (A_{k\ell})_{1 \leq \ell \leq k-1} \in \mathcal{X}_k = \{0, 1\}^{k-1}$. The graph H can be recovered from $X = (X_2, \dots, X_n) \in \mathcal{X} = \times_{k=2}^n \mathcal{X}_k$. Moreover, for some functions $F, F_v : \mathcal{X} \rightarrow [0, 1]$, $\tau(G, v) = F_v(X)$ and

$$F(X) = \sum_{v=1}^n F_v(X).$$

Let $X, X' \in \mathcal{X}$ and assume that $X'_\ell = X_\ell$ unless $k = \ell$ for some $1 \leq k \leq n$. Then, since τ is h -local, we have $F_v(X) = F_v(X')$ unless v is within graph distance (in G) at most h from vertex k . By definition, there are $N_h(G, k)$ such vertices. It follows that

$$|F(X) - F(X')| \leq N_h(G, k).$$

It remains to apply Azuma-Hoeffding's inequality. □

5.3 Proof of Theorem 9

Proof of (i), step one : control of the resolvent. The beginning of the argument repeats the proof of Theorem 7, we simply replace the Lebesgue measure ℓ by $\mathbb{P} \otimes \ell$. Let $S = [-d, d]$, B be the adjacency operator of $\text{perc}(\Gamma, p)$, $G_o(z) = \langle e_o, (B - zI)^{-1} e_o \rangle$ and $h_\eta(\lambda) = \frac{1}{\pi} \Im G_o(\lambda + i\eta)$. Let f be the random density of the absolutely continuous part of $\mu_o = \mu_{\text{perc}(\Gamma, p)}^{e_o}$. We have that $\mathbb{P} \otimes \ell$ -a.s. $f(\lambda) = \lim_{\eta \rightarrow 0} h_\eta(\lambda)$. By assumption, $m = \mathbb{E} \int_S f(\lambda) d\lambda = \mathbb{E}(\mu_o)_{ac}(\mathbb{R})$ is positive and for any $\varepsilon > 0$, we can find a pair (a, b) of positive numbers satisfying,

$$\int_S \mathbb{E} f(\lambda) \mathbf{1}(a \leq f(\lambda) \leq b) d\lambda \geq m - \varepsilon/2.$$

Arguing as in Theorem 7, there exist positive constants a', b', q, η_0 such that, if

$$\bar{h}_\eta(\lambda) = \mathbb{E} [h_\eta(\lambda) \mathbf{1}(a' \leq h_t(\lambda) \leq b', \text{ for all } t \in [0, 1])],$$

the Borel set $K_1 = \{\lambda \in S : \bar{h}_\eta(\lambda) > q \text{ for all } \eta \in [0, 1]\}$, satisfies for any $0 \leq \eta \leq \eta_0$,

$$\int_{K_1} \bar{h}_\eta(\lambda) d\lambda \geq m - \varepsilon. \quad (22)$$

We now use our concentration lemma to deduce from the above inequality an inequality satisfied by the resolvent of $\text{perc}(G, p)$. Without loss of generality, we may assume that δ given by (4) is smaller than $(a' \wedge q)/4$ and that $\eta_1 = 20d \log(2h)/h \leq \eta_0$. If A is the adjacency matrix of $\text{perc}(G, p)$ and $v \in V(G)$, we set $\tilde{h}_{\eta, v}(\lambda) = \frac{1}{\pi} \Im(A - z)_{vv}^{-1}$, with $z = \lambda + i\eta$. We also set $\delta_0 = 1/(\zeta h \pi d)$ and

$$\tau_{\eta, v}(\lambda) = \frac{1}{\pi} \Im(A_v - z)_{vv}^{-1} \mathbf{1}_{\mathcal{E}_v(\lambda)},$$

where A_v is the adjacency operator of the graph $(G, v)_h$ and $\mathcal{E}_v(\lambda)$ denotes the event,

$$\mathcal{E}_v(\lambda) = \left\{ a' - \delta_0 \leq \frac{1}{\pi} \Im(A_v - (\lambda + it))_{vv}^{-1} \leq b' + \delta_0, \text{ for all } t \in [\eta_1, 1] \right\}.$$

First, if $v \notin B(h)$, then by Corollary 25, if $\eta_1 \leq \eta \leq 1$, $\mathbb{E} \tau_{\eta, v}(\lambda) \geq \bar{h}_\eta(\lambda) - \delta_0$. Note that $(G, v) \mapsto \tau_{\eta, v}(\lambda)$ is a h -local functional in the sense defined above Lemma 30 and it is bounded by $1/(\pi \eta_1) \leq (2h)/(\zeta d \pi)$. We deduce from this lemma that if, $t = 2\delta/(\zeta \pi d)$ and $\lambda \in \mathbb{R}$,

$$\frac{1}{n} \sum_{v=1}^n \tau_{\eta, v}(\lambda) \geq \bar{h}_\eta(\lambda) - \delta_0 - \frac{2h}{\zeta \pi d} \frac{B(h)}{n} - t \geq \bar{h}_\eta(\lambda) - \frac{5\delta}{\zeta \pi d}, \quad (23)$$

with probability at least $1 - \exp(-n\delta^2/(2h^2 M_h^2(G)))$. As a consequence of Corollary 25, if (23) holds, then

$$\frac{1}{n} \sum_{v=1}^n \tilde{h}_{\eta, v}(\lambda) \mathbf{1}_{\tilde{\mathcal{E}}_v(\lambda)} \geq \bar{h}_\eta(\lambda) - \frac{\delta}{\pi d},$$

where $\tilde{\mathcal{E}}_v(\lambda) \supset \mathcal{E}_v(\lambda)$ denotes the event,

$$\tilde{\mathcal{E}}_v(\lambda) = \left\{ a' - 2\delta_0 \leq \tilde{h}_{t,v}(\lambda) \leq b' + 2\delta_0, \text{ for all } t \in [\eta_1, 1] \right\}.$$

We may use a net argument as in Theorem 7. We consider a $\delta\eta_1^2$ -net of $(8d+2)/(\delta\eta_1^2) \leq h^2/\delta$ points on boundary of the rectangle $R = \{z : \eta_1 \leq \Im(z) \leq 1, \Re(z) \in [-2d, 2d]\}$. From the union bound and the maximum principle,

$$\inf_{(\lambda+i\eta) \in R} \frac{1}{n} \sum_{v=1}^n \tilde{h}_{\eta,v}(\lambda) \mathbf{1}_{\tilde{\mathcal{E}}'_v(\lambda)} - \bar{h}_\eta(\lambda) \geq -\frac{3\delta}{4\pi d} - \frac{\delta}{\pi} \geq -\delta, \quad (24)$$

with probability at least $1 - h^2\delta^{-1} \exp(-n\delta^2/(2h^2M_h^2(G)))$. In the above expression, $\tilde{\mathcal{E}}'_v(\lambda)$ is defined as $\tilde{\mathcal{E}}_v(\lambda)$ with $2\delta_0$ replaced by $2\delta_0 + \delta/\pi$. We set $a'' = a' - 2\delta_0 - \delta/\pi$ and $b'' = b + 2\delta_0 + \delta/\pi$.

Proof of (i), step two : from resolvent to eigenvectors. We may now use the above inequality to find delocalized eigenvectors in the finite graph $\text{perc}(G, p)$. For $\lambda \in \mathbb{R}$, let

$$V(\lambda) = \{v : \tilde{\mathcal{E}}'_v(\lambda) \text{ holds}\} = \{v : a'' \leq \tilde{h}_{\eta,v}(\lambda) \leq b'' \text{ for all } \eta_1 \leq \eta \leq 1\}.$$

If (24) holds and $\lambda \in K_1$, then $|V(\lambda)| \geq (q - \delta)n/b'' \geq (q/(2b''))n$ with q introduced above (22). Notably, if (24) holds, by Corollary 27, we find that for any $\lambda \in K_1$, if $v \in V(\lambda)$, $\eta = c_0(\log h)/h$ and $I = [\lambda - \eta, \lambda + \eta]$,

$$c_1\eta \leq \sum_{k \in \Lambda_I} \psi_k(v)^2 \leq c_2\eta, \quad (25)$$

for some positive constants c_0, c_1, c_2 . We sum over all $v \in V(h)$ and set $c_3 = c_1q/(2b'')$, we deduce that

$$c_3\eta n \leq c_1\eta |V(\lambda)| \leq \sum_{v \in V(\lambda)} \sum_{k \in \Lambda_I} \psi_k(v)^2 \leq |\Lambda_I|. \quad (26)$$

On the other end, consider $L \subset K_1$ a maximal 2η -separated set, that is for any $\lambda \neq \lambda'$ in L , $|\lambda - \lambda'| \geq 2\eta$ and L has maximal cardinal. Then, by maximality, $K_1 \subset \cup_{\lambda \in L} (\lambda - 2\eta, \lambda + 2\eta)$, and it implies that $|L| \geq \ell(K_1)/(4\eta)$. Also, the set $\Lambda = \cup_{\lambda \in L} \Lambda_{(\lambda-\eta, \lambda+\eta)}$ is a disjoint union. Since $|\Lambda| \leq n$, from the pigeon hole principle, we deduce that there exists a subset $L^* \subset L$ of cardinal at least $|L|/2 \geq \ell(K_1)/(8\eta)$ such that for all $\lambda \in L^*$,

$$|\Lambda_{(\lambda-\eta, \lambda+\eta)}| \leq \frac{2n}{|L|} \leq \frac{8}{\ell(K_1)}\eta n.$$

We now prove that if $\lambda \in L^*$, then a positive proportion of the eigenvectors in $\Lambda_{(\lambda-\eta, \lambda+\eta)}$ have a positive proportion of their norm supported on $V(\lambda)$. To this end we apply the inequality (25) to each $\Lambda_{(\lambda-\eta, \lambda+\eta)}$ with $\lambda \in L^*$. We find that if (24) holds and $I = (\lambda - \eta, \lambda + \eta)$ then

$$\frac{1}{|\Lambda_I|} \sum_{v \in V(\lambda)} \sum_{k \in \Lambda_I} \psi_k(v)^2 \geq c_1 \frac{\eta |V(\lambda)|}{|\Lambda_I|} \geq c_4,$$

where $c_4 = c_1 q \ell(K_1)/(2^4 b'')$. For $k \in \Lambda_I$, let $x_k = \sum_{v \in V(\lambda)} \psi_k(v)^2$. For $0 < t < 1$, if $\Lambda_I(t) = \{k \in \Lambda_I : x_k > t\}$, we observe that, since $x_k \leq 1$,

$$c_4 |\Lambda_I| \leq \sum_{k \in \Lambda_I} x_k \leq |\Lambda_I(t)| + t(|\Lambda_I| - |\Lambda_I(t)|).$$

We deduce that $|\Lambda_I(t)| \geq |\Lambda_I|(c_4 - t)/(1 - t)$. If $0 < t < c_4$, $\Lambda_I(t)$ is a positive proportion of Λ_I and from (26),

$$\left| \bigcup_{\lambda \in L^*} \Lambda_{(\lambda - \eta, \lambda + \eta)}(t) \right| \geq \frac{1 - c_4}{1 - t} |L^*| c_3 \eta n \geq \alpha n,$$

with $\alpha = ((1 - c_4)/(1 - t))(\ell(K_1)/8)c_3$.

Finally, if $k \in \Lambda_{[\lambda - \eta, \lambda + \eta]}$ then (25) implies that for any $v \in V(\lambda)$, $\psi_k^2(v) \leq c_2 \eta$. It thus concludes the proof of the first part of Theorem 9 with $\rho = t$ and a new constant $c_1 = \sqrt{c_0 c_2}$.

Proof of (ii). Let $\alpha = \rho = 1 - \sqrt{\beta}$, $\beta = 4\pi(1 - m')$ and $m' = m - \varepsilon - \delta$. By Corollary 27 (see (19)), it is sufficient to prove (up to adjusting the constant c) that, if (24) holds, there are at least αn eigenvalues λ_k , such that there exists a set V_k and a real l_k with $|l_k - \lambda_k| \leq c\eta$, $y_k = \sum_{v \in V_k} \psi_k(v)^2 \geq \rho$ and for all $v \in V_k$, $\tilde{h}_{\eta, v}(l_k) \leq c$.

First, from (22), if (24) holds, we have

$$\int_{K_1} \frac{1}{n} \sum_{v \in V(\lambda)} \tilde{h}_{\eta, v}(\lambda) d\lambda \geq m'.$$

We start with a regularization of the sets $V(\lambda)$. To this end, we consider the open set $U = \bigcup_{\lambda \in L} (\lambda - 3\eta, \lambda + 3\eta) \supset \bar{K}_1$ with L as above. We can find a finite partition $(P_l)_l$ of U , $U = \bigcup_l P_l$, such that P_l is an interval of length at most $\pi\eta^2$ and no eigenvalue lies on the boundary ∂P_l . For each l , we consider an element $x_l \in P_l$ and define, $V_l = \{v \in V : \tilde{h}_{\eta, v}(x_l) \leq b'' + 1\}$. Then, since $|\tilde{h}_{\eta, v}(x) - \tilde{h}_{\eta, v}(y)| \leq |x - y|\eta^{-2}\pi^{-1}$, we find for all $\lambda \in P_l$, $V(\lambda) \subset V_l$ and

$$\sum_l \frac{1}{n} \sum_{v \in V_l} \int_{P_l} \tilde{h}_{\eta, v}(\lambda) d\lambda \geq m'.$$

Since $\int \frac{1}{n} \sum_{v=1}^n \tilde{h}_{\eta, v}(\lambda) d\lambda = 1$, we get

$$\sum_l \frac{1}{n} \sum_{v \notin V_l} \int_{P_l} \tilde{h}_{\eta, v}(\lambda) d\lambda \leq 1 - m'.$$

Now, we observe that $\sum_{v \notin V_l} \tilde{h}_{\eta, v}(\lambda)$ is equal to $\Im(g_{\mu_l}(z))/\pi$ with $z = \lambda + i\eta$, $\mu_l = \sum_k (1 - y_{k, l}) \delta_{\lambda_k}$ and

$$y_{k, l} = \sum_{v \in V_l} \psi_k(v)^2 = 1 - \sum_{v \notin V_l} \psi_k(v)^2.$$

If $k \in \Lambda_U$, then $k \in \Lambda_{P_l}$ for a unique l , and we set $y_k = y_{k,l}$. From Lemma 28 and the assumption $\mu_l(\partial P_l) = 0$, we find

$$\sum_{k \in \Lambda_U} (1 - y_k) = \sum_l \sum_{k \in \Lambda_{P_l}} (1 - y_{k,l}) = \sum_l \mu_l(P_l) \leq n2\pi(1 - m').$$

Similarly, if (24) holds,

$$\int_{K_1^c} \frac{1}{n} \sum_{v=1}^n \tilde{h}_{\eta,v}(\lambda) d\lambda \leq 1 - m'.$$

Then, we apply Lemma 28 to the open set $(\bar{K}_1)^c \subset K_1^c$. Since $U^c \subset (\bar{K}_1)^c$, we find

$$|\Lambda_{U^c}| \leq n2\pi(1 - m').$$

Hence, we have checked that, if (24) holds,

$$\sum_{k \in \Lambda_U} y_k \geq n(1 - \beta).$$

For $0 < t < 1$, if $n_t = \{k \in \Lambda_U : y_k > t\}$, we have $n(1 - \beta) \leq (n - n_t)t + n_t$. Hence,

$$n_t \geq \frac{1 - \beta - t}{1 - t} n = \alpha n.$$

We choose $t = \rho = 1 - \sqrt{\beta}$, we get $\alpha = 1 - \sqrt{\beta}$. It concludes the proof of the theorem.

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